数理解析研究所講究録
784巻 1992年 53-73

THE CARTESIAN CLOSEDNESS OF THE CATEGORY Fuzz AND FUNCTION SPACES ON TOPOLOGICAL FUZZES

Zhongqiang YANG

ABSTRACT. In this paper, the cartesian closedness of the category Fuzz consisting of all fuzzes and all order homomorphisms is proved. Moreover, in the exponent of the category we set up topologies of pointwise convergence and uniform convergence.

1980 Math. Subj. Class(1985 Revision):54A40, 54B30, 54C35, 06D10.
Key words: fuzz, cartesian closed, function space, completely distributive lattice, order homomorphism.

THE CARTESIAN CLOSEDNESS OF THE CATEGORY Fuzz AND FUNCTION SPACES ON TOPOLOGICAL FUZZES

A fuzz is a pair (F, ') of a completely distributive lattice F and an anti-order involution ': F⟶F. The category Fuzz is consisted by all fuzzes and all mappings called order homomorphisms (see §1). In a fuzz a generalized topology has been defined and dicussed [1][3]. Moreover, G.J.Wang defined a generalized topology on a completely distributive lattice. In [8], the author set up function spaces on the skelton. In this paper, we hope to define function spaces in topological fuzz. For this purpose, we have to consider which fuzz the function space is set up on. We find that the exponent in the category Fuzz is proper to set up function space. Hence, it is necessary to prove that the category is cartesian closed.

We give needed definitions and theorems in section 1. After an auxiliary concept is defined and studied in section 2, the cartesian closedness of the category Fuzz is proved in section 3. At last section we set up topologies of pointwise convergence and uniform convergence.

§1. Preliminary

For a poset P and $A \subset P$, let $\downarrow A = \{x \in P: x \leq a$ for some

$a\in A\}$. For $x\in P$, let $\downarrow x=\downarrow\{x\}$. Dual, we have $\uparrow A$ and $\uparrow x$.

Definition 1.1. A fuzz [1] is a pair $(F, ')$ of a completely distributive lattice F and an anti-order involution $':F\longrightarrow F$, that is , $x\leq y$ if and only if $y'\leq x'$ and $x''=x$ for all $x,y\in F$.

An order homomorphism [3] $f:(F, ')\longrightarrow(G, ')$ from $(F, ')$ to $(G, ')$ is a mapping $f:F\longrightarrow G$ such that f preserves arbitrary joins and f^{-1} preserves $'$, where $f^{-1}:G\longrightarrow F$ is defined as

$$f^{-1}(b)=\vee\{a\in F: f(a)\leq b\}$$

for all $b\in G$.

Let Fuzz be the category of all fuzzes and all order homomorphisms.

Let L be a complete lattice and $x,y\in L$. If for every $A\subset L$, $\vee A\geq y$ implies $a\geq x$ for some $a\in A$, then it is denoted by $x\lhd y$ [6]. Let $\beta(y)=\{x\in L: x\lhd y\}$. It is trivial that $\beta(0)=\phi$.

For a complete lattice L and $m\in L$, m is called molecular [3] if $m\neq 0$ and $m\leq a\vee b$ implies $m\leq a$ or $m\leq b$. The set of all molecules in L is denoted by $M(L)$.

Lemma 1.2. [6] A complete lattice L is a completely distributive lattice if and only if $x=\vee\beta(x)$ for every $x\in L$.

Lemma 1.3. For a completely distributive lattice L, we have

(i). [6] $x\lhd y$ implies $x\lhd z\lhd y$ for some $z\in L$;

(ii).[7] $x=\vee\{m\in M(L): m\leq x\}=\vee\{m\in M(L): m\lhd x\}$.

Lemma 1.4.[4] For every order homomorphism f, we have that f^{-1} preserves arbitrary joins and arbitrary meets.

Definition 1.5. [1][3] A topological fuzz is a triple $(F, ', \delta)$ such that $(F, ')$ is a fuzz and $\delta \subset F$ satisfies:

(1). $0, 1 \in \delta$;

(2). δ is closed for finite meets;

(3). δ is closed for arbitrary joins.

Definition 1.6. [5] A category C is called cartesian closed if

(1). C has a termial;

(2). C has finite products;

(3). C has exponents, that is, for every pair of objects A, B, there is an object B^A and a morphism $ev: B^A \times A \longrightarrow B$ (called an exponent for A and B and the evalution, resepectively) such that for every object C and morphism $f: C \times A \longrightarrow B$ there is an unique morphism $\Lambda(f): C \longrightarrow B^A$ such that $f = ev \circ (\Lambda(f) \times id_A)$.

Other undefined terms can found in [3].

§2. Parallelisms and Products

It is very surprising that only to consider all order homomorphisms is not enough for studying the cartesian closedness of the category Fuzz. Hence, we do not assume that all mappings studied in this paper are order homomorphisms but , in fact, we only assume that all mappings preserve arbitrary joins. For fuzzes F and G, let $[F\longrightarrow G]$ be the set of all mappings preserving arbitrary joins, moreover, with the pointwise order, $[F\longrightarrow G]$ is a complete lattice.

Definition 2.1. Let $f,g\in[F\longrightarrow G]$. If for every $x\in F$ the following formula

$$f^{-1}((g(x))')\leq x'$$

holds, then we call that f and g are paralle and denote by $f\| g$, otherwise by $f\nparallel g$.

Proposition 2.2. Let $f,g,h\in[F\longrightarrow G]$. We have that

(1). if $f\| g$, then $g\| f$;

(2). if $f\| g$ and $f\leq h$, then $h\| g$.

Proof. (1). Let $x\in F$ and $x_1=g^{-1}((f(x))')$. Then $g(x_1)\leq(f(x))'$ and hence $f(x)\leq(g(x_1))'$. It follows that $x\leq f^{-1}((g(x_1)')\leq x_1'$ because $f\| g$. So we have $g^{-1}((f(x))')=x_1\leq x'$. Thus $g\| f$.

(2). Trivial.

Proposition 2.3. If $f:F\longrightarrow G$ is an order homomorphism and $g: F\longrightarrow G$ preserves arbitrary joins, then $f\| g$ if and only if $f\leq g$.

Proof. $f\| g \leftrightarrow f^{-1}((g(x))')\leq x'$ for all $x\in F$

$\leftrightarrow x\leq(f^{-1}((g(x))'))'=f^{-1}(g(x))$ for all $x\in F$

$\leftrightarrow f(x)\leq g(x)$ for all $x\in F$

$\leftrightarrow f\leq g$.

Corollary 2.4. If f and g are order homomorphisms, then $f \| g$ if and only if $f=g$.

B.Hutton have proved that there are arbitrary poducts in the category Fuzz [1]. To prove the cartesian closedness of the category, we give another forms of the products. Of course, there exists a natural isomorphism between the two products.

Let $\{F_s : s\in S\}$ be a family of fuzzes and $\prod_{s\in S} F_s$ the directed product. Suppose that

$$\bigotimes_{s\in S} F_s=\{A\subset \prod_{s\in S} F_s :\ A=\downarrow A \text{ and for every } x\in A \text{ there exists } y\in A \text{ such that } x_s \lhd y_s \text{ for every } s\in S\}.$$

Then $\bigotimes_{s\in S} F_s$ is a completely distributive lattice [8]. Moreover, for all $x\in \prod_{s\in S} F_s$, we have $\prod_{s\in S}\beta(x_s)\in \bigotimes_{s\in S} F_s$ and all elements of $\bigotimes_{s\in S} F_s$ are unions of some forms of $\prod_{s\in S}\beta(x_s)$. Now for $A\in \bigotimes_{s\in S} F_s$, let

$$A'=\wedge\{ \bigcup_{s\in S}\beta(x_s')\times \prod_{t\neq s}\beta(1_t) :\ \prod_{s\in S}\beta(x_s)\subset A\}. \tag{2.2}$$

Then we have

Theorem 2.5. $(\bigotimes_{s\in S} F_s,\ ')$ is a product of the family $\{F_s : s\in S\}$ in the category Fuzz.

Proof. We only prove that $'$ is an involution, the other details are omitted. (*cf*. [8][1])

Let $A\in \bigotimes_{s\in S} F_s$ and $\prod_{s\in S}\beta(x_s)\subset A$. Then for every $\prod_{s\in S}\beta(y_s)\subset A'$, we have

$$\prod_{s\in S}\beta(y_s)\subset \bigcup_{s\in S}(\beta(x_s')\times \prod_{t\neq s}\beta(1_t))$$

and hence

$$\prod_{s\in S}\beta(x_s)\subset\bigcup_{s\in S}(\beta(y_s')\times\prod_{t\neq s}\beta(1_t)).$$

Thus $\prod_{s\in S}\beta(x_s)\subset A''$, it follows $A\subset A''$. Conversely, if $A''\subset A$ does not hold, then there exists $x\in A''$ such that $\prod_{s\in S}\beta(x_s)\not\subset A$, that is, there exists $y\in\prod_{s\in S}\beta(x_s)\setminus A$. Hence, for any $a\in A$, there is $s(a)\in S$ such that $y_{s(a)}\leq a_{s(a)}$, that is, $a_{s(a)}'\leq y_{s(a)}'$. It follows that there exists $b_{s(a)}^a\in\beta(a_{s(a)}')$ such that $b_{s(a)}^a\leq y_{s(a)}'$. Now let

$$d_s=\begin{cases}\wedge\{b_{s(a)}^a:\ s(a)=s\}, & \text{if } s=s(a) \text{ for some } a\in A;\\ 1, & \text{if } s\neq s(a) \text{ for all } a\in A.\end{cases}$$

for all $s\in S$. Then we have $d_s\leq x_s'$ for every $s\in S$. Thus

$$\prod_{s\in S}\beta(d_s)\not\subset(\prod_{s\in S}\beta(x_s))' \tag{2.3}$$

But on other hand, for every $a\in A$ we have that $d_{s(a)}\leq b_{s(a)}^a\leq a_{s(a)}'$ and hence

$$\prod_{s\in S}\beta(d_s)\subset A'. \tag{2.4}$$

(2.2), (2.3), (2.4) and $\prod_{s\in S}\beta(x_s)\subset A$ are contrary. It is followed that $A''\subset A$.

In the next section we need the following lemmas. At first, for an order homomorphism $f:H\otimes F\longrightarrow G$ and $z\in H$, we define $f_z:F\longrightarrow G$ as following

$$f_z(x)=f(\beta(z)\times\beta(x)). \tag{2.5}$$

Then $f_z\in[F\longrightarrow G]$ but it is not necessarily an order homomorphism.

Lemma 2.6. If $f:H\otimes F\longrightarrow G$ is an order homomorphism, then for every pair z_1, $z_2\in H$, $z_1\leq z_2'$ implies $f_{z_1}\Vert f_{z_2}$.

Proof. For $x_1\in F$, let $x_2=f_{z_2}^{-1}((f_{z_1}(x_1))')$. Then $f_{z_2}(x_2)\leq$

$(f_{z_1}(x_1))'$, that is,

$$f(\beta(z_2)\times\beta(x_2))\leq(f(\beta(z_1)\times\beta(x_1)))'.$$

Hence, from f being an order homomorphism, we have

$$\beta(z_2)\times\beta(x_2)\leq(\beta(z_1')\times\beta(1))\cup(\beta(1)\times\beta(x_1')).$$

Thus $x_2\leq x_1'$ because $z_2\leq z_1'$. It is followed that $f_{z_2}\|f_{z_1}$.

Lemma 2.7. Let $f:H\otimes F\longrightarrow G$ be an order homomorphism, $A\subset[F\longrightarrow G]$ and $r,z\in H$ satisfing $r\in\beta(z)$. If $f_r\not\leq k$ for all $k\in A$, then there exists $r_0\in H$ such that $r_0\leq z'$ and $f_{r_0}\not\|k$ for all $k\in A$.

Proof. For every $k\in A$, $f_r\not\leq k$ implies $f_r(x_k)\leq k(x_k)$ for some $x_k\in F$. Hence, $f(\beta(r)\times\beta(x_k))\not\leq k(x_k)$. From f being an order homomorphism, it is followed that

$$f^{-1}((k(x_k)')\leq(\beta(r)\times\beta(x_k))'=(\beta(r')\times\beta(1))\cup(\beta(1)\times\beta(x_k')).$$

Thus there exists $(z_k,a_k)\in H\times F$ such that

$$z_k\leq r',\ a_k\leq x_k'$$

but

$$f(\beta(z_k)\times\beta(a_k))\leq(k(x_k))'. \tag{2.6}$$

Let $r_0=\bigwedge_{k\in A}z_k$. Then $r_0\leq z'$ because otherwise we have $z\leq\bigvee_{k\in A}z_k'$ and hence $r\leq z_k'$ for some $k\in A$. Moreover (2.6) implies $f_{r_0}(a_k)\leq(k(x_k))'$. Thus

$$f_{r_0}^{-1}((k(x_k))')\geq a_k\leq x_k'$$

and hence $f_{r_0}\not\|k$ for every $k\in A$.

§3. The Cartesian Closedness of the Category Fuzz

Let F, G and H be fuzzes and $f:H\otimes F\longrightarrow G$ an order homomorphism. For $z\in H$, set

$$A(f,\ z)=\{h\in[F\longrightarrow G]:\ h\leq f_r \ \text{ for some } \ r\in\beta(z)\}.$$

Let G^F be the smallest family of subsets of $[F\longrightarrow G]$ which is closed for arbitrary unions and containes all forms $A(f,\ z)$ for any order homomorphism $f:H\otimes F\longrightarrow G$ and $z\in H$. In G^F, we define $':G^F\longrightarrow G^F$ as following:

$$A'=\bigcup\{B\in G^F:\ f\not\leq g \ \text{ for all } f\in A \ \text{ and } \ g\in B\}$$

for $A\in G^F$.

Lemma 3.1. G^F as a subfamily of the family of all subsets of $[F\longrightarrow G]$ is a complete lattice and for $A,B\in G^F$, $A\lhd B$ if and only if there exists a fuzz H and an order homomorphism $f:H\otimes F\longrightarrow G$, $z,r\in H$ such that $r\in\beta(z)$ and $A\subset A(f,r)\subset A(f,z)\subset B$.

Proof. It is followed from the definitions of G^F and $\lhd$.

Lemma 3.2. $(G^F,\ ')$ is a fuzz.

Proof. By the last lemma and Lemma 1.2, it is easy to show that G^F is a completely distributive lattice. To prove that $';G^F\longrightarrow G^F$ is an anti-order ivolution, we have only to verify $B'\leq A'$ for $A\leq B$ and $A''=A$ for all $A,B\in G^F$. The former is trivial, moreover, it is followed from Proposition 2.2 that $A\subset A''$. Thus, the remainer is to show that $A''\subset A$. Otherwise, there exists $h\in[F\longrightarrow G]$ such that $h\in A''\setminus A$ and hence there exists a fuzz H

and $r, z \in H$, order homomorphism $f: H \otimes F \longrightarrow G$ such that $h \le f_r$ and $r \in \beta(z)$, $f_r \in A(f, z) \subset A''$ but $f_r \notin A$. By Lemma 1.3, there exists $r_1 \in H$ such that $r \lhd r_1 \lhd z$. Moreover, it is followed from Lemma 2.7 that there exists $z_0 \in H$ such that $z_0 \le r_1'$ and $f_{z_0} \not\geq k$ for all $k \in A$. Thus $A(f, z_0) \subset A'$ from the definition. Furthermore, $A(f, z) \subset A''$ implies that $f_{r_1} \not\geq f_{z_1}$ for all $z_1 \lhd z_0$ and hence by Proposition 2.6 we have that $z_1 \le r_1'$ for all $z_1 \lhd z_0$. It follows that $z_0 \le r_1'$. Contrary!

Lemma 3.3. For fuzzes F, G, define $ev: G^F \otimes F \longrightarrow G$ as following

$$ev(C) = \vee\{f(x) : f \in A \in G^F \text{ and } \beta(A) \times \beta(x) \subset C\}.$$

Then ev is an order homomorphism.

Proof. Clearly, ev preserves arbitrary joins. Now, we prove ev^{-1} preserves $'$, that is , for every $y \in G$, we have

$$ev^{-1}(y') = (ev^{-1}(y))'. \qquad (3.2)$$

Firstly, we prove that for all $A_1, A_2 \in G^F$ and $x_1, x_2 \in F$, if $\beta(A_1) \times \beta(x_1) \subset ev^{-1}(y)$ and $\beta(A_2) \times \beta(x_2) \subset ev^{-1}(y')$ then

$$\beta(A_1) \times \beta(x_1) \subset (\beta(A_2) \times \beta(x_2))' \qquad (3.3)$$

In fact, in case $x_1 \le x_2'$, (3.3) is trivial. In case $x_1 \le x_2'$, for any $f_1 \in A_1$ and $f_2 \in A_2$, we have that $f_1(x_1) \le y$ and $f_2(x_2) \le y'$. Hence $x_1 \le f_1^{-1}(y) \le f_1^{-1}((f_2(x_2))')$. Thus $f_1^{-1}((f_2(x_2))') \le x_2'$, that is, $f_1 \not\geq f_2$. It is followed that $A_1 \subset A_2'$ and hence (3.3) holds. Moreover, from the fact above proved it is followed that $ev^{-1}(y') \le (ev^{-1}(y))'$.

Secondly, we prove that $ev^{-1}(y') \ge (ev^{-1}(y))'$. If $A \in G^F$

and $x \in F$ satisfy that $\beta(A) \times \beta(x) \leq ev^{-1}(y')$, then there exist $h \in A$ and $x_1 \in \beta(x)$ such that $h(x_1) \leq y'$. Withought lost generalized, we suppose that $h_{z_1} = f$ and $A(f, z) \subset A$ for some order homomorphism $f: H \otimes F \longrightarrow G$ and z_1, $z \in H$ with $z_1 \in \beta(z)$. Thus we have

$$f^{-1}(y) \leq (\beta(z_1) \times \beta(x_1))' = (\beta(z_1') \times \beta(1)) \cup (\beta(1) \times \beta(x_1')).$$

Hence there exists $(z_2, x_2) \in H \times F$ such that

$$\beta(z_2) \times \beta(x_2) \leq f^{-1}(y) \quad \text{but} \quad z_2 \leq z_1', \quad x_2 \leq x_1'. \tag{3.4}$$

Let $B = A(f, z_2)$. Then $B \not\subset A'$. In fact, $z_2 \leq z_1'$ implies that there exists $z_3 \in \beta(z_2)$ such that $z_3 \leq z_1'$ and hence $f_{z_3} \| f_{z_1}$. Because $f_{z_3} \in B$ and $f_{z_1} \in A$ we have $B \not\subset A'$. Hence from $x_2 \leq x_1'$ it is followed that

$$\beta(B) \times \beta(x_2) \not\subset (\beta(A) \times \beta(x_1))'.$$

But on the other hand, (3.4) implies

$$\beta(B) \times \beta(x_2) \subset ev^{-1}(y).$$

Hence we have

$$\beta(A) \times \beta(x_1) \not\subset (ev^{-1}(y))'.$$

Thus

$$ev^{-1}(y') \geq (ev^{-1}(y))'.$$

(3.2) is proved.

Lemma 3.4. For an order homomorphism $f: H \otimes F \longrightarrow G$, define $\Lambda(f): H \longrightarrow G^F$ as

$$\Lambda(f)(z) = A(f, z). \tag{3.5}$$

Then

(1). $\Lambda(f)$ is an order homomorphism;

(2). $ev\circ(\Lambda(f)\otimes id_F)=f$;

(3). $\Lambda(f)$ is an unique order homomorphism satifying (2).

Proof. From the definition we have that $\Lambda(f)$ preserves arbitrary joins. Now, suppose that $A\in G^F$, $z_1=\Lambda(f)^{-1}(A)$, $z_2=\Lambda(f)^{-1}(A')$. Then $\Lambda(f)(z_1)\leq A$, $\Lambda(f)(z_2)\leq A'$. Hence for all $r_1\lhd z_1$ and $r_2\lhd z_2$ we have $f_{r_1}\not\leq f_{r_2}$ and hence $r_1\leq r_2'$. Thus we have $z_1\leq z_2'$, that is,

$$\Lambda(f)^{-1}(A')\leq(\Lambda(f)^{-1}(A))' \qquad (3.6)$$

Conversely, if $\Lambda(f)^{-1}(A')\not\geq(\Lambda(f)^{-1}(A))'$, then there exists $z\in H$ such that $z\leq(\Lambda(f)^{-1}(A))'$ but $z\not\leq\Lambda(f)^{-1}(A')$. Hence, $\Lambda(f)(z)\not\leq A'$. It follows that there exists r_1, $r\in H$ such that $f_{r_1}\notin A'$ and $r_1\lhd r\lhd z$. Lemma 2.7 implies that there exists $z_1\in H$ such that $z_1\leq r'$ and $f_{z_1}\not\leq k$ for all $k\in A'$. Hence we have $A(f,z_1)\subset A''=A$, that is, $\Lambda(f)(z_1)\subset A$. Thus, $z_1'\geq(\Lambda(f)^{-1}(A))'\geq z\geq r$, which is contrary to $z_1'\not\geq r$. Hence

$$(\Lambda(f)^{-1}(A))'\leq\Lambda(f)^{-1}(A').$$

(1) is completed.

(2). Let $(z, x)\in H\otimes F$. We have that

$$\begin{aligned}
&ev\circ(\Lambda(f)\otimes id_F)(\beta(z)\times\beta(x))\\
&=ev(\beta(\Lambda(f)(z))\times\beta(x))\\
&=ev(\beta(A(f\ ,z))\times\beta(x))\\
&=\vee\{f_r(x):\ r\in\beta(z)\}\\
&=\vee\{f(\beta(r)\times\beta(x):\ r\in\beta(z)\}\\
&=f(\beta(z)\times\beta(x)).
\end{aligned}$$

Hence, because both of $ev\circ(\Lambda(f)\otimes id_F)$ and f preserve arbitrary joins and the definition of product we have

$$ev\circ(\Lambda(f)\otimes id_F)=f.$$

(3). We have only to prove that if $g_1, g_2: H\longrightarrow G^F$ are order homomorphisms and $ev\circ(g_1\otimes id_F)=ev\circ(g_2\otimes id_f)$ then $g_2\le g_1$. Let z_1, $z_2\in H$ and $z_1\in\beta(z_2)$. Then $g_1(z_1)\in\beta(g_1(z_2))$ and hence, by Lemma 3.1, there exists an order homomorphism $h:E\otimes F\longrightarrow G$ and $e\in E$, $e_1\in\beta(e)$ such that $g_1(z_1)\le\downarrow h_{e_1}$ and $A(h, e)\le g_1(z_2)$. So we have that for $x\in F$,

$$\begin{aligned}&ev\circ(g_2\otimes id_F)(\beta(z_1)\times\beta(x))\\ &=ev\circ(g_1\otimes id_F)(\beta(z_1)\times\beta(x))\\ &=ev(\beta(g_1(z_1)\times\beta(x))\\ &\le h_{e_1}(x).\end{aligned}$$

Thus, $g_2(z_1)\le\downarrow h_{e_1}\le g_1(z_2)$. It follows that $g_2\le g_1$ because $z_1\lhd z_2$ are arbitrary and g_2 preserves arbitrary joins.

As a conclusion we have

Theorem 3.5. The category Fuzz is cartesian closed.

§4. Topologies of Pointwise Convergence and Uniform Convergence

To study function spaces on topological fuzz, we introduce a concept of subspaces of a topological fuzz.

Let F be a fuzz. We consider a mapping $j:F\longrightarrow F$ which satisfies the following conditions

(S1). $j(a)\leq a$ for all $a\in F$;

(S2). $j\circ j=j$;

(S3). j presvering arbitrary joins;

(S4). $j((j(a))')=j(a')$ for all $a\in F$.

Let $F_j=\{j(a):\ a\in F\}$. Then F_j is closed with arbitrary joins and arbitrary meets and hence is a completely distrbutive lattice. Moreover, $(F_j,\ c)$, where $a^c=j(a')$, is a fuzz and the embeding mapping $i:F_j\longrightarrow F$ is an order-homomorphism.

Furthermore, if $(F,\ \delta)$ is a topological fuzz, then $(F_j,\ \delta_j)$, where $\delta_j=j(\delta)$, is also topological fuzz.

Definition 4.1. If $j;F\longrightarrow F$ satisfies the conditions (S1)-(S4), then F_j is called a subfuzz of F. $(F_j,\ \delta_j)$ is called a topological subfuzz of $(F,\ \delta)$.

On subfuzz and topological subfuzz, we will discuss in another paper, In here, we only discuss a special subfuzz —— a subfuzz of G^F for topological fuzzes F and G consisting of continuous mappings.

Let $(F,\ \delta)$ and $(G,\ \varepsilon)$ be topological fuzzes. Define $j:G^F\longrightarrow G^F$ as follows:

$$j(A)=\bigcup\{A(f.z):\ f:H\otimes F\longrightarrow G \text{ is a continuous order homomorphism and } H \text{ is a topological fuzz, } z\in H \text{ and } A(f,\ z)\subset A\} \tag{4.1}$$

Lemma 4.2. $j:G^F\longrightarrow G^F$ satisfies the conditions (S1)-(S4).

Proof. (S1) and (S2) are trivial. To show (S3) we have only to note that $A(f,z)=\bigcup\{A(f,r): r\in\beta(z)\}$ and if $A(f,z)\subset\bigcup_{s\in S}A_s$ then for every $r\in\beta(z)$, there exists $s\in S$ such that $A(f, r)\subset A_s$. At last, we prove (S4). Clearly, $j(j(A)')\geq j(A')$ for all $A\in G^F$. To show $j(j(A)')\leq j(A')$ we have only to verify that for a topological fuzz H and a continuous order homomorphism $f:H\otimes F\longrightarrow G$, $z\in H$, if $A(f, z)\subset(j(A))'$ then $A(f, z)\subset A'$. In fact, if $A(f, z)\not\subset A'$, then there exists $r\in H$ such that $r\in\beta(z)$ but $f_r\in A'$. By Lemma 2.7, there exists $z_0\in H$ such that $z_0\not\leq z'$ and $f_{z_0}\not\geq k$ for all $k\in A'$. Hence from the definition it is followed that $A(f, z_0)\subset A''=A$. Hence, by the definition of j, we have $A(f, z_0)\subset j(A)$. Thus, from $A(f, z)\subset(j(A))'$ and Lemma 2.6, we have $z_0\leq z'$. Contradition!

Definition 4.3. For $j:G^F\longrightarrow G^F$ defined in (4.1), the subfuzz $G^F{}_j$ is called the fuzz of continuous mappings (between F and G) and denoted by CG^F.

Definition 4.4. Let (F, δ) and (G, ε) be topological fuzzes. For every $m\in M(F)$ and closed element c in (G, ε), set up

$$C(m, c)=\bigcup\{A\in G^F: f(m)\leq c \text{ for all } f\in A\}. \tag{4.2}$$

Then the topology on G^F generated by

$$\{C(m, c): m\in M(F) \text{ and } c \text{ is a closed in } (G, \varepsilon)\}$$

as a cosubbase is called a topology of pointwise convergence.

For the topology of pointwise convergence, we can prove the following results. The proofs of the results are similar to those in [8] and hence are omitted.

Theorem 4.5. Let $\{f_\sigma : F \longrightarrow G,\ \sigma \in \Sigma\}$ be a net consisting of order homomorphisms and $f : F \longrightarrow G$ an order homorphism. Then $\{\downarrow f_\sigma,\ \sigma \in \Sigma\}$ converges to $\downarrow f$ in G^F with respect to the topology of pointwise convergence if and only if $\{f_\sigma(m),\ \sigma \in \Sigma\}$ convergens $f(m)$ for every $m \in M(F)$.

Theorem 4.6. Let F, G, H be topological fuzzes. Then for every continuous order homomorphism $f : H \otimes F \longrightarrow G$, we have $\Lambda(f) : H \longrightarrow CG^F$ is continuous with respect to the topology of pointwise convergence.

Now we define a topology of (quasi-)uniform convergence on function space. An (quasi-)uniformity is defined by Hutton in 1977 [2].

Let G be a fuzz. A mapping $g : G \longrightarrow G$ is called valu-increase if $g(b) \geq b$ for all $b \in G$. Let $V(G)$ be the of all valu-increase mappings from G to itself preserving arbitrary joins. Then $V(G)$ is a complete lattice with respect to pointwise order. Let $f \wedge g$ be the greatest lower bounded of $\{f,\ g\}$ in the lattice. It is easy that $(f \wedge g)(b) \leq f(b) \wedge g(b)$ for all $b \in G$. But equality is not necessary. For every $g \in V(G)$ we define $g_{-1} : G \longrightarrow G$ as following

$$g_{-1}(b) = \wedge\{c \in G :\ g(c') \leq b'\}. \qquad (4.3)$$

Then $g_{-1} \in V(G)$.

Definition 4.7.[2] Let G be a fuzz, $\phi\neq D\subset V(G)$ is called a quasi-uniformity on G if

(1). For every $d\in D$ and $e\in V(G)$, if $d\leq e$ then $e\in D$;

(2). If $d, e\in D$, then $d\wedge e\in D$;

(3). For every $d\in D$, there exists $e\in D$ such that $e\circ e\leq d$.

Furthermore, if D also satisfies (4), then D is called an uniformity on G:

(4). If $d\in D$, then $d_{-1}\in D$.

Let F and G be fuzzes and D an (quasi-)uniformity on G. For every $d\in D$, define $\hat{d}:G^F\longrightarrow G^F$ as following:

$$\hat{d}(A)=\bigcup\{B\in G^F:\text{ For every } f\in B \text{ there exists } g\in A \text{ such that } f\leq d\circ g\}. \tag{4.4}$$

Then $\hat{d}\in V(G^F)$. (Sometime $\hat{d}$ is denoted by $(d)^\wedge$)

Lemma 4.8. For $f,g\in[F\longrightarrow G]$ and $d\in V(G)$, $f\| d\circ g$ if and only if $d_{-1}\circ f\| g$.

Proof. Suppose $f\| d\circ g$. Let $x\in F$ and $x_1=g^{-1}((d_{-1}\circ f(x))')$. Then

$$g(x_1)\leq(d_{-1}(f(x)))'=\vee\{b':\ d(b')\leq(f(x))'\}$$

and hence

$$d\circ g(x_1)\leq\vee\{d(b'):\ d(b')\leq(f(x))'\}\leq(f(x))'.$$

Thus, by $f\| d\circ g$, we have

$$x\leq f^{-1}((d\circ g(x_1))')\leq x_1'.$$

That is, $g^{-1}(((d_{-1}\circ f)(x))')\leq x'$ for all $x\in F$, i.e., $g\| d_{-1}\circ f$.

"Only if " is followed from $(d_{-1})_{-1}=d$.

Lemma 4.9. $\hat{}$ has the following properties:

(1). $\hat{d}\wedge\hat{e}\geq(d\wedge e)\hat{}$;

(2). $\hat{d}\circ\hat{d}\leq(d\circ d)\hat{}$;

(3). $(\hat{d})_{-1}\geq(d_{-1})\hat{}$

for $d, e\in V(G)$.

Proof. (1) and (2) are trivial.

(3). For $A\in G^F$, we have

$(d_{-1})\hat{}(A)=\cup\{B\in G^F:\ B$ satisfies the following (4.5)$\}$,

$(\hat{d})_{-1}(A)=\wedge\{C\in G^F:\ C$ satifies the following (4.6)$\}$.

To complete (3), we have only to prove that if B, $C\in G^F$ satisfy, respectively,

$$\text{for every } h\in B,\text{ there is } k\in A \text{ such that } h\leq d_{-1}\circ k, \tag{4.5}$$

$$\hat{d}(C')\leq A' \tag{4.6}$$

then $B\subset C$. Suppose that $f:H\otimes F\longrightarrow G$ is an order homomorphism and $r, z\in H$. If $r\in\beta(z)$ and $f_r\in C$ then, by Lemma 2.7. there exists $z_0\in H$ such that $z_0\leq z'$ and $f_{z_0}\not\leq k$ for all $k\in C$. Hence, $A(f, z_0)\subset C'$. By (4.6), it is followed that there exists $r_0\in\beta(z_0)$ such that $r_0\leq z'$ and $d\circ f_{r_0}\in A'$. Then $f_z\in B$ and hence $B\subset C$. In fact, otherwise, there exists $k\in A$ such that $f_z\leq d_{-1}\circ k$. It is followed from Proposition 2.2 and Lemma 2.6 that $d_{-1}\circ k\| f_{r_0}$. By Lemma 4.8 we have $k\| d\circ f_{r_0}$, which contradit with $d\circ f_{r_0}\in A'$ and $k\in A$.

Theorem 4.10. If F, G are fuzzes and D is an (quasi-) uniformity then

$$\hat{D}=\uparrow\{\hat{d}:\ d\in D\} \tag{4.7}$$

is an (quasi-)uniformity.

Proof. It is immediately obtained from last lemma.

Theorem 4.11. If F is a topological fuzz and (G, D) is a quasi-unifomity then evolution $ev: CG^F \otimes F \longrightarrow G$ is continuous with respect to the topology of quasi-uniform convergence on CG^F.

Proof. The proof is similar to those in [8] and is omitted.

Lemma 4.12. If $f,g:F\longrightarrow G$ are order homomorphisms and $d\in V(G)$, then $f\leq d\circ g$ if and only if $g\leq d_{-1}\circ f$.

Proof. If $g\not\leq d_{-1}\circ f$, then $g(a)\not\leq d_{-1}(f(a))$ for some $a\in F$. That is, $g(a)\not\leq\wedge\{b\in G:\ d(b')\leq(f(a))'\}$. Thus there exists $b\in G$ such that $d(b')\leq(f(a))'$ but $g(a)\not\leq b$. Because f and g are order homomorphisms we have $f(g^{-1}(b'))\not\leq(d\circ g)(g^{-1}(b'))$. Therefore $f\not\leq d\circ g$.

Theorem 4.13. Let F be a topological fuzz and (G, D) an uniformity. If a net $\{\downarrow f_\sigma:\ \sigma\in\Sigma\}$ consisting of continuous order homomorphisms from F into G converges to $\downarrow f$, where $f:F\longrightarrow G$ is an order homomorphism, with respect to the topology of uniform convergence in G^F, then f is continuous.

Proof. Let c be closed in G and $p\in M(F)$. If $p\le f^{-1}(c)$, then there exists $d\in D$ such that

$$f(p)\le d(c). \tag{4.8}$$

Choose $e\in D$ such that

$$e\circ e\circ e\le d \quad \text{and} \quad e_{-1}=e \tag{4.9}$$

Set

$$A=\bigcup\{B\in G^F:\ f\le e\circ h \ \text{ for all } \ h\in B\}. \tag{4.10}$$

Then $\downarrow f\le\hat{e}(A)$ and hence $\downarrow f\le\bar{A}$, where $(\bar{\ })$ is the closure operator in a topological fuzz been considering. Thus $\downarrow f_\sigma\le\bar{A}$ for some $\sigma\in\Sigma$. It follows from (4.10) that

$$f\le e\circ f_\sigma. \tag{4.11}$$

Therefore, (4.8) and (4.9) imply that $f_\sigma(p)\le(e\circ e)(c)$. Thus

$$p\le f_\sigma^{-1}(\overline{e(c)}). \tag{4.12}$$

From (4.9), (4.11) and Lemma 4.12 it is followed that

$$f^{-1}(c)\le f_\sigma^{-1}(e(c))\le f_\sigma^{-1}(\overline{e(c)}).$$

Hence by (4.12) and the continuouity of f_σ we have $p\le\overline{f^{-1}(c)}$. Thus $f^{-1}(c)=\overline{f^{-1}(c)}$, that is $f^{-1}(c)$ is closed.

Acknowledgement. I would like to express my gratitude to Professor G.J.Wang and Professor R.Nakagawa for their directions.

REFERENCES

1 . B.Hutton, Produts of fuzzy topological spaces, Topology Appl., 11(1980), 59-67.

2 . ——, Uniformities on fuzzy topological spaces, J.Math.Anal. Appl., 58(1977), 557-571.

3 . G.J.Wang, Generalized topologically molecular lattices, Seientia Sinica(Series A), 27(1984),No.8, 785-798.

4 . ——, An introduction to L-fuzzy topology, Shaanxi Normal Univ. Publishing Co., 1988. (in Chinese)

5 . A.Jung, Cartesian closed categories of domains, ph. D. Thesis, Thchich Hochschule Darmstadt, 1988.

6 . G.Bruns, Distributivität und subdireket zerlegbarkeit vollständiger verbände Archiv d. Math., Vol.12(1961), 61-66.

7 . M.Kamara, Treillis continus et treillis completement distributifs, Semigroup Formm, Vol. 16(1978), 387-388.

8 . Z.Q.Yang, The cartesian closedness of the category CDL and function spaces on topological CDLs, preprint.

Institute of Mathematics
University of Tsukuba
Tsukuba-shi,Ibaraki, 305
Japan

Departmenet of Mathematics
Shaanxi Normal University
Xi'an, 710062
China